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SECONDARY FLOWS IN A PLANE CHANNEL

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Formation of the secondary self-oscillating modes of flow branching away from the Poiseuille flow in a plane channel, is investigated. Conditions of appearance of the secondary self-oscillating modes at nearly critical values of the Reynolds number were obtained earlier in [1,2]. It was shown, in particular, that the existence of the secondary flows can be established by analyzing the linearized equations only.

In the present paper the formation of secondary flows in a plane channel, periodic in x and t, is studied with help of the asymptotic solutions of the Orr — Sommerfeld equations. It is proved that in the sufficiently small neighborhood of almost every value of the Reynolds number R lying on the neutral curve of the linear theory of stability, values of R exist such that the Navier — Stokes equations have solutions periodic in x and t.

1. Let us consider a flow of a viscous incompressible fluid in a plane unbounded channel. We choose the coordinate system in such a manner that the x-axis coincides with the channel axis and the y-coordinates of the walls are $+1$ and -1.

The dimensionless equation of the stream function ψ has the form

$$\frac{\partial \Delta\psi}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y} - \frac{1}{R}\Delta^2\psi = 0 \tag{1.1}$$

where the mean efflux velocity is chosen as the characteristic velocity. The stream function must satisfy the boundary conditions

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = 0 \quad \text{for} \quad y = \pm 1 \tag{1.2}$$

When the values of R are sufficiently small, the problem (1.1), (1.2) has a unique stationary solution $\psi_0(y)$ such that

$$d\psi_0/dy = U(y) \equiv {}^3/_2 (1 - y^2) \tag{1.3}$$

We shall seek solutions different from (1.3) and periodic in x and t. Let us set $\zeta = x - ct$ and write the stream function in the form

$$\psi = \psi_0(y) + \frac{1}{R}\Phi(\zeta, y) \tag{1.4}$$

where $\Phi(\zeta, y)$ is $2\pi/\alpha_0$-periodic in ζ. Substituting (1.4) into (1.1), (1.2), we obtain

$$R\left[(U-c)\frac{\partial\bar\Delta\Phi}{\partial\zeta} - U''\frac{\partial\Phi}{\partial\zeta}\right] - \bar\Delta^2\Phi = \frac{\partial\Phi}{\partial\zeta}\frac{\partial\bar\Delta\Phi}{\partial y} - \frac{\partial\Phi}{\partial y}\frac{\partial\bar\Delta\Phi}{\partial\zeta}, \quad \bar\Delta = \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial\zeta^2}\right) \tag{1.5}$$

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial \zeta} = 0 \quad \text{for} \quad y = \pm 1 \tag{1.6}$$

Since Φ is a periodic function of ζ and the flow of fluid across the transverse section of the channel can be assumed constant, (1.6) can be replaced by the following boundary conditions:

$$\Phi = \frac{\partial \Phi}{\partial y} = 0 \quad \text{for} \quad y = \pm 1 \tag{1.7}$$

Let us consider the linearized problem

$$R\left[(U - c)\frac{\partial \bar{\Delta}\varphi}{\partial \zeta} - U'' \frac{\partial \varphi}{\partial \zeta}\right] - \bar{\Delta}^2\varphi = 0 \tag{1.8}$$

where φ should be $2\pi / \alpha_0$ -periodic in ζ and satisfy the conditions (1.7). The boundary value problem (1.8), (1.7) has solutions of the form

$$\varphi = f(y)e^{i\alpha\zeta} + f^*(y)e^{-i\alpha\zeta}, \qquad \alpha = k\alpha_0 \tag{1.9}$$

where $f(y)$ is a solution of the boundary value problem for the Orr — Sommerfeld equation

$$i\alpha R\left[(U - c)\left(\frac{d^2}{dy^2} - \alpha^2\right)f - U''f\right] - \left(\frac{d^2}{dy^2} - \alpha^2\right)^2 f = 0 \tag{1.10}$$

$$f(\pm 1) = f'(\pm 1) = 0 \tag{1.11}$$

Since the problem (1.10), (1.11) is symmetrical in y, it can be solved for the even and odd functions $f(y)$ separately. To find the even eigenfunctions, we can replace the conditions (1.11) by

$$f(-1) = f'(-1) = f'(0) = f'''(0) = 0 \tag{1.12}$$

The values $R = R_0(\alpha)$ and $c = c_0(\alpha)$ for which the problem (1.10), (1.12) has nontrivial solutions, is found from the equation

$$D_1(\alpha, R, c) = 0 \tag{1.13}$$

where $D_1(\alpha, Rc)$ is the characteristic determinant of the problem (1.10), (1.12).

According to [3], there exists a solution $R_0(\alpha)$ of (1.13) describing in the plane (α, R) a curve which we shall call neutral. This solution is such that for the values of α and R belonging to this curve, an eigenvalue $c = c_0(\alpha, R_0)$ of the problem (1.10), (1.12), exists.

Let us consider the problem (1.10), (1.12) for the values of parameters lying on the upper branch of the neutral curve and such, that (see [3])

$$\alpha \to 0, \quad R_0 = O(\alpha^{-11}), \quad c_0 = O(\alpha^2) \tag{1.14}$$

It was proved in [4] that the asymptotic solutions of the Orr — Sommerfeld equation determining the eigenvalues of the problem, can be used within this range of the parameter values.

The fundamental system of solutions of (1.10) can be constructed for two "smooth" solutions approaching the solutions of the degenerate equation as $\alpha R \to \infty$, and from two solutions of boundary layer type. The solutions obtained by Heisenberg (see [3]) are used as two linearly independent solutions of the degenerate equation.

Using the asymptotic solutions obtained in [5], we can show that the fundamental system of solutions of (1.10) can be written in the form

$$\frac{d^m}{dy^m}\varphi_1(y)=\frac{d^m}{dy^m}\varphi_1^{(0)}(y)+O\left(\frac{1}{\lambda^2}\right)+O(\alpha^2c^3P_m) \tag{1.15}$$

$$\frac{d^m}{dy^m}\varphi_2(y)=\frac{d^m}{dy^m}\varphi_2^{(0)}(y)+O\left(\frac{1}{\lambda^2 c}\right)+P_m$$

$$\frac{d^m}{dy^m}\varphi_3(y)=\frac{d^m}{dy^m}(U-c)^{-5/4}e^{-\lambda Q}\left[1+O\left(\frac{1}{\lambda Q}\right)\right]$$

$$\frac{d^m}{dy^m}\varphi_4(y)=\frac{d^m}{dy^m}(U-c)^{-5/4}e^{\lambda Q}\left[1+O\left(\frac{1}{\lambda Q}\right)\right]$$

$$m=0,1,2,3,\quad \lambda=\sqrt{\alpha R},\quad P_0=O(\lambda^{-2}z^{-2}),\quad P_1=O(\lambda^{-1/4}z^{-2})$$

$$P_2=o(z^{-1}),\quad P_3=o(z^{-2}),\quad z=\left(1-\frac{2}{3}c\right)^{-1/2}(y-y_c)$$

$$y_c=-\left(1-\frac{2}{3}c\right)^{1/2},\quad Q=\int_{y_c}^{y}\sqrt{i(U-c)}\,dy$$

If $|z|\geqslant z_0>0$, then $P_m=O(\lambda^{-2})$ and $\varphi_1^{(0)}$, $\varphi_2^{(0)}$ are solutions of the degenerate equation

$$\varphi_{1,2}^{(0)}=(U-c)\,(q_{1,2}^{(0)}+\alpha^2 q_{1,2}^{(1)}+\ldots) \tag{1.16}$$

$$q_1^{(0)}=1,\quad q_2^{(0)}=\int_{-1}^{y}(U-c)^{-2}dy,\quad q_{1,2}^{(n+1)}=\int_{-1}^{y}(U-c)^{-2}dy\int_{-1}^{y}(U-c)^2 q_{1,2}^{(n)}dy$$

It was shown in [4] that two even, linearly independent solutions exist : the smooth solution f_1, and the boundary layer solution f_2 (p is any positive number)

$$\frac{d^m}{dy^m}f_1=\frac{d^m}{dy^m}(\varphi_1^{(0)}-k\varphi_2^{(0)})+O(\alpha^2P_m)+O\left(\frac{\alpha^2}{\lambda^2 c}\right) \tag{1.17}$$

$$k=\alpha^2\int_{-1}^{0}(U-c)^2dy\,[1+O(\alpha^2)]$$

$$\frac{d^m}{dy^m}f_2=\frac{d^m}{dy^m}\varphi_3+O(\lambda^{-p}),\quad m=0,1,2,3 \tag{1.18}$$

The eigenfunction of the problem (1.10), (1.12) is represented by a linear combination of the solutions of (1.17) and (1.18) satisfying the first two conditions of (1.12). A given eigenvalue has a single corresponding eigenfunction.

We shall prove that in the region (1.14) the problem (1.10), (1.12) has no associated functions. To do this we must show that

$$\int_{-1}^{1}\omega^*\left(\frac{d^2}{dy^2}-\alpha^2\right)f\,dy=2\int_{-1}^{0}\omega^*\left(\frac{d^2}{dy^2}-\alpha^2\right)f\,dy\neq 0 \tag{1.19}$$

where ω is a solution of the problem conjugate to (1.10), (1.12). The function ω^* satisfies the equation

$$i\alpha R\left[(U-c)\left(\frac{d^2}{dy^2}-\alpha^2\right)\omega^* + 2U'\frac{d\omega^*}{dy}\right]-\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\omega^* = 0 \qquad (1.20)$$

and the boundary conditions (1.12).

Since ω^* and f are analytic functions, the integration path in (1.19) can be accomodated into the complex y-plane

$$\int_{-1}^{0}\omega^*\left(\frac{d^2}{dy^2}-\alpha^2\right)f\,dy = \int_{\Gamma}\omega^*\left(\frac{d^2}{dy^2}-\alpha^2\right)f\,dy \qquad (1.21)$$

$$\Gamma = \{y;|\, y + {}^1/_2\,| = {}^1/_2,\ \operatorname{Im} y \leqslant 0\}$$

and the asymptotic expressions (1.15) hold on the semicircle Γ [5].

The solutions of (1.20) can be written in terms of the solutions of (1.10)(see [6]). Using (1.15) we can show that the fundamental system of solutions of (1.20) can be written in the form

$$\frac{d^n}{dy^n}\chi_1(y) = \frac{d^n}{dy^n}\frac{\varphi_1^{(0)}}{(U-c)}[1+O(\xi^{-1})]+O\left(\frac{1}{\xi^{2-n}}\right) \qquad (1.22)$$

$$\frac{d^n}{dy^n}\chi_2(y) = \frac{d^n}{dy^n}\frac{\varphi_2^{(0)}}{(U-c)}[1+O(\xi^{-1})]+O\left(\frac{1}{c\xi^{2-n}}\right)$$

$$\frac{d^n}{dy^n}\chi_3(y) = \frac{d^n}{dy^n}(U-c)^{-1/4}e^{-\lambda Q}[1+O(\xi^{-1})]$$

$$\frac{d^n}{dy^n}\chi_4(y) = \frac{d^n}{dy^n}(U-c)^{-1/4}e^{\lambda Q}[1+O(\xi^{-1})]$$

$$n = 0,1;\ \xi = \lambda Q$$

where the estimates are written for y lying on the curve Γ.

The eigenfunctions f and ω^* can be represented in the form

$$f = \frac{1}{f_2'(-1)}f_2(y) - \frac{1}{f_1'(-1)}f_1(y) \qquad (1.23)$$

$$\omega^* = \frac{1}{\omega_2^{*\prime}(-1)}\omega_2^*(y) - \frac{1}{\omega_1^{*\prime}(-1)}\omega_1^*(y)$$

where $f_1(y)$ and $f_2(y)$ are defined by the expressions (1.17) and (1.18), and $\omega_1^*(y)$ is a smooth, even solution of (1.20)

$$\frac{d^n}{dy^n}\omega_1^*(y) = \frac{d^n}{dy^n}\left\{\frac{\varphi_1^{(0)}}{U-c}[1+O(\xi^{-1})] - k\frac{\varphi_2^{(0)}}{U-c}[1+O(\xi^{-1})]\right\}+O\left(\frac{1}{\xi^{2-n}}\right)$$

$n = 0,1;$ p is any positive number and $\omega_2^*(y)$ is the even boundary layer solution

$$\omega_2^*(y) = \chi_3(y) + O(\lambda^{-p})$$

Substituting (1.23) into (1.21) we can show, that

$$\int_{-1}^{1} \omega^* \left(\frac{d^2}{dy^2}-\alpha^2\right) f dy = 2\int_{\Gamma} \omega^* \left(\frac{d^2}{dy^2}-\alpha^2\right) f dy = \frac{2}{3} c\,[1+O(\alpha^2)] \qquad (1.24)$$

where the integral was estimated with help of the relations (1.14).

In the region where the neutral curve satisfies the conditions (1.14), the multiplicity of the eigenvalues of the problem (1.10), (1.12) is equal to one.

To find the odd eigenfunctions, we can replace the conditions (1.1) by

$$f(-1) = f'(-1) = f(0) = f''(0) = 0 \qquad (1.25)$$

Let $D_2(\alpha, R, c)$ be the characteristic determinant of the problem (1.10), (1.25). Using the asymptotic expressions for $D_1(\alpha, R, c)$ and $D_2(\alpha, R, c)$ (see [7]) we can show that when α and R lie on the upper branch of the neutral curve in the region (1.14) and c has values satisfying the equations (1.13), then $D_2(\alpha, R, C) \neq 0$. Consequently the eigenvalue of the problem (1.10), (1.11) has multiplicity of one in the region occupied by the neutral curve (1.14). The multiplicity of the eigenvalue $c = c(\alpha, R)$ coincides with the zero multiplicity of the characteristic determinant at the point $c = c(\alpha, R)$.

Let $D(\alpha, R, c)$ be the characteristic determinant of the problem (1.10), (1.11). The eigenvalues of multiplicity greater than one should satisfy the following system of equations:

$$D(\alpha, R, c) = 0, \quad \partial D(\alpha, R, c)/\partial C = 0 \qquad (1.26)$$

The system (1.26) is equivalent to the system composed of four real equations for determining three real unknowns α, R and c. It was proved that a segment of the curve defined by the first equation of the system (1.26) exists, on which $\partial D / \partial C \neq 0$.

Using the property of analyticity of $D(\alpha, R, c)$ in α, R and c, we can show that the system (1.26) can have solutions only at isolated points of the neutral curve. The following lemma holds: the Orr — Sommerfeld problem for the Poiseuille flow has simple eigenvalues almost everywhere on the neutral curve.

2. Let us consider the problem (1.5), (1.7). We shall seek small solutions in the form [2] ($\varepsilon > 0$ is the amplitude of the solution)

$$\Phi(\zeta, y) = \varepsilon\varphi(\zeta, y) + v(\zeta, y), \int_{-1}^{1}\int_{0}^{2\pi/\alpha} e^{-i\alpha\zeta} v \left(\frac{d^2}{dy^2}-\alpha^2\right) \omega^* d\zeta dy = 0 \qquad (2.1)$$

The solution $\varphi(\zeta, y)$ of the linearized problem has the form (1.9), and $\omega^*(y)$ denotes the solution of the problem (1.20), (1.11). Let us set

$$v = \sum_{n=2}^{\infty} \varepsilon^n v_n, \quad R = \sum_{n=0}^{\infty} \varepsilon^n R_n, \quad Rc = R_0 \sum_{n=0}^{\infty} \varepsilon^n c_n \qquad (2.2)$$

Substituting (2.2) into (1.5), we obtain the following infinite system of equations for v_n:

$$R_0\left[(U-c_0)\frac{\partial\bar{\Delta}v_n}{\partial\zeta}-U''\frac{\partial v_n}{\partial\zeta}\right]-\bar{\Delta}^2 v_n=R_0\sum_{l=1}^{n-2}c_l\frac{\partial\bar{\Delta}v_{n-l}}{\partial\zeta}- \tag{2.3}$$

$$\sum_{l=1}^{n-2}R_l\left(U\frac{\partial\bar{\Delta}v_{n-l}}{\partial\zeta}-U''\frac{\partial v_{n-l}}{\partial\zeta}\right)+\sum_{l=2}^{n-1}K(v_l,v_{n-l})+\delta_{n2}K(\varphi,\varphi)+$$

$$K(v_{n-1},\varphi)+K(\varphi,v_{n-1})+R_0c_{n-1}\frac{\partial\bar{\Delta}\varphi}{\partial\zeta}-R_{n-1}\left(U\frac{\partial\bar{\Delta}\varphi}{\partial\zeta}-U''\frac{\partial\varphi}{\partial\zeta}\right)$$

$$K(u,v)=\frac{\partial u}{\partial\zeta}\frac{\partial\bar{\Delta}v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial\bar{\Delta}v}{\partial\zeta},\qquad \delta_{ni}=\begin{cases}1, & n=i\\ 0, & n\neq i\end{cases}$$

The functions v_n must satisfy the boundary conditions (1.7) and be $2\pi/\alpha$ - periodic in ζ, and the coefficients R_0 and c_0 can be found from

$$D(\alpha,R_0,c_0)=0 \tag{2.4}$$

When $n\geqslant 1$, the quantities R_n and c_n are found from the condition that the equations (2.3) have a solution.

Let us denote by Z_1 the set of all values of α from which $D(\alpha,R_0,c_0)$ has a zero of order higher than first. By virtue of the lemma given above the set Z_1 is not more than denumerable.

Let Z_2 denote the set of such α that the equations

$$D(m\alpha,R_0,c_0)=0,\qquad m=2,3,\ldots \tag{2.5}$$

hold simultaneously with (2.4) for at least one value of m. It can be shown that Z_2 is not more than denumerable.

In what follows, we shall assume that

$$\alpha\,\overline{\in}\,Z_1\cup Z_2 \tag{2.6}$$

Let us consider the equation (2.3) for $n=2$. Since α satisfies (2.6), the condition of its solvability can be written in the form

$$R_0c_1-R_1\,I_2/I_1=0 \tag{2.7}$$

$$I_1=\int_{-1}^{1}\omega^*\left(\frac{d^2}{dy^2}-\alpha^2\right)f\,dy,\qquad I_2=\int_{-1}^{1}\omega^*\left[U\left(\frac{d^2}{dy^2}-\alpha^2\right)f-U''f\right]dy$$

Let us obtain the value of I_2/I_1 for α and R_0 lying on the upper branch of the neutral curve in the region (1.14). The Orr — Sommerfeld equation yields

$$\frac{I_2}{I_1}=c_0+\frac{1}{i\alpha R_0I_1}\int_{-1}^{1}\omega^*\left(\frac{d^2}{dy^2}-\alpha^2\right)^2 f\,dy \tag{2.8}$$

Using the asymptotic expressions (1.23) for f and ω^*, we obtain

$$\int_{-1}^{1}\omega^*\left(\frac{d^2}{dy^2}-\alpha^2\right)^2 f\,dy=-e^{-i\pi/4}\sqrt{\alpha R_0c_0}\,[1+O(\alpha^{-1/2}R_0^{-1/2}c_0^{-3/2})] \tag{2.9}$$

and substitution of (2.9) and (1.24) into the right-hand side of (2.8) we find

$$\operatorname{Im}\frac{I_2}{I_1}=\frac{3}{2\sqrt{2}}\frac{1}{\sqrt{\alpha R_0 c_0}}\,[1+O(\alpha^2)]\neq 0 \tag{2.10}$$

Now we shall show that $\operatorname{Im}(I_1 / I_2)$ can vanish only at isolated points of the neutral curve. We shall assume the opposite and write, that

$$\operatorname{Im}(I_2 / I_1) = 0 \tag{2.11}$$

holds in some region of the neutral curve. It can be shown that (2.11) is equivalent to the equation

$$M(\alpha, R_0, c_0) = 0 \tag{2.12}$$

where $M(\alpha, R_0, c_0)$ is a real analytic function of α, R_0 and c_0. Then by virtue of the analyticity of $D(\alpha, R_0, c_0)$ and $M(\alpha, R_0, c_0)$, Eq. (2.11) will hold on the whole neutral curve, and this contradicts (2.10).

Let Z_3 be the set of all α for which (2.11) holds. In what follows we shall assume that

$$\alpha \equiv Z_1 \cup Z_2 \cup Z_3 \tag{2.13}$$

Then from (2.7) we find that $c_1 = R_1 = 0$. The function v_2 has the form

$$v_2(\zeta, y) = v_{20}(y) + v_{22}(y)\, e^{2i\alpha\zeta} + v_{22}^*(y)\, e^{-2i\alpha\zeta} \tag{2.14}$$

The coefficients c_2 and R_2 can be found from the condition that (2.3) has a solution when $n = 3$

$$i\alpha R_0 c_2 I_1 - i\alpha R_2 I_2 + \frac{\alpha_0}{2\pi}\int_{-1}^{1}\int_{0}^{2\pi/\alpha_0} \omega^*(y)\, e^{-i\alpha\zeta}\,[K(v_2, \varphi) + K(\varphi, v_2)]\, d\zeta dy = 0 \tag{2.15}$$

Since α satisfies (2.13), Eq. (2.15) has a solution in R_2 and c_2. The remaining R_n and c_n can be found in a similar manner.

Condition (2.13) coincides with the conditions of the theorem (2.1) of [1]. The following theorem can be formulated: almost all values of R lying on the neutral curve of the problem (1.10), (1.12) represent a branch point of the cycle for the problem (1.5), (1.7).

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